Periodicity in thermal physics

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We describe some examples of objects in thermal physics that exhibit periodicity in Euclidean time. We discuss the path integral construction of the partition function, the periodicity of thermal correlators and the trick to compute black hole temperatures.

1 Partition function

An important object in thermal physics is the partition function for the theory at temperature $T = 1/\beta$. We can write the classical definition for the case of a quantum system as

$$Z = \sum_n e^{-\beta E_n} = \sum_n \langle n| \, e^{-\beta H} \, |n\rangle = \mathrm{Tr}\big(e^{-\beta H}\big) \,. \tag{1}$$

It is of course possible to take the trace over the Hilbert space of the theory in any basis, so we can use the position basis $|X\rangle$ instead:

$$Z = \int dX \, \langle X| \, e^{-\beta H} \, |X\rangle \,. \tag{2}$$

To go further in the calculation of the partition function, we must remember that the path integral in quantum mechanics is an object that computes transition amplitudes in the following way:

$$\langle x_f, t_f | x_i, t_i \rangle = \langle x_f| \, e^{-iH(t_f - t_i)} \, |x_i\rangle = \int_{\substack{x(t_i)=x_i \\ x(t_f)=x_f}} \mathcal{D}x(t) \, e^{iS[x]} \,. \tag{3}$$

This is the propagator to go from position x_i at time t_i to position x_f at time t_f. The path integral is basically a sum over all paths $x(t)$ that satisfy the boundary conditions, weighted by their action $S[x] = \int_{t_i}^{t_f} dt \, L(x,t)$. To connect this to the partition function, we must study the path integral in Euclidean/imaginary time by defining $\tau = it \Rightarrow t = -i\tau$. We then have

$$\langle x_f| \, e^{-H(\tau_f - \tau_i)} \, |x_i\rangle = \int_{\substack{x(\tau_i)=x_i \\ x(\tau_f)=x_f}} \tilde{\mathcal{D}}x(\tau) \, e^{-S_E[x]} \,, \tag{4}$$

where the Euclidean action is $S_E[x] = \int_{\tau_i}^{\tau_f} d\tau \, L(x, -i\tau)$. The main difference in the Euclidean case is the Lagrangian that is used. If, for example, the usual Lagrangian is $L(x,t) = \frac{m}{2}\left(\frac{dx}{dt}\right)^2$, then the one that is used in the Euclidean case is $L(x, -i\tau) = -\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2$. Using this, it is now clear that we can write

$$\langle X| \, e^{-\beta H} \, |X\rangle = \int_{x(0)=x(\beta)=X} \tilde{\mathcal{D}}x(\tau) \, e^{-S_E[x]} \,. \tag{5}$$

Integrating over X, we recover the partition function. However, the effect of adding this integral is simply to allow for any possible trajectory that is periodic with period β in the path integral. Thus the result is

$$Z = \int dX \int_{x(0)=x(\beta)=X} \tilde{\mathcal{D}}x(\tau) \, e^{-S_E[x]} = \int_{\substack{periodic\ x(\tau) \\ 0 \le \tau \le \beta}} \tilde{\mathcal{D}}x(\tau) \, e^{-S_E[x]} \,. \tag{6}$$

In a sense we can say that the partition function if the Euclidean path integral where we integrate over all functions (not necessarily smooth) on a circle of circumference β.

2 Thermal correlators (KMS condition)

Another object of importance in thermal field theories is correlation functions. At thermal equilibrium, a system is in a state of energy E with probability $\frac{e^{-\beta E}}{Z}$. This means that we can compute the thermal expectation value of quantum operators in the following way:

$$\langle \mathcal{O} \rangle_\beta = \sum_i \frac{e^{-\beta E_i}}{Z} \langle E_i | \mathcal{O} | E_i \rangle = \frac{1}{Z} \operatorname{Tr}\big(e^{-\beta H} \mathcal{O}\big) . \tag{7}$$

Now we will use this method to compute the thermal two-point function of a field, which is basically the thermal Green's function (maybe up to a sign),

$$G_\beta(\tau, x) \equiv \langle \mathcal{O}(\tau, x) \mathcal{O}(0,0) \rangle_\beta = \frac{1}{Z} \operatorname{Tr}\big(e^{-\beta H} \mathcal{O}(\tau, x) \mathcal{O}(0,0)\big) . \tag{8}$$

Note that, as in any QFT correlator, there is an implicit time ordering (here Euclidean time ordering) that puts operators in chronological order from right to left. The usual time evolution in quantum mechanics can be translated to Euclidean time:

$$\mathcal{O}(t, x) = e^{iHt} \mathcal{O}(0, x) e^{-iHt} \Longrightarrow \mathcal{O}(\tau, x) = e^{H\tau} \mathcal{O}(0, x) e^{-H\tau} \tag{9}$$

and this can be used to translate one of the operators from τ to $\tau - \beta$:

$$G_\beta(\tau, x) = \frac{1}{Z} \operatorname{Tr}\big(\mathcal{O}(\tau - \beta, x) e^{-\beta H} \mathcal{O}(0,0)\big) . \tag{10}$$

Using cyclicity of the trace, we can rewrite this as

$$G_\beta(\tau, x) = \frac{1}{Z} \operatorname{Tr}\big(e^{-\beta H} \mathcal{O}(0,0) \mathcal{O}(\tau - \beta, x)\big) \tag{11}$$

and because of the time ordering this is equivalent to

$$G_\beta(\tau, x) = G_\beta(\tau - \beta, x) . \tag{12}$$

This shows that the thermal two-point function is periodic with period β.

3 Black hole temperature

In the 70s, Stephen Hawking showed that black holes are thermal object and calculated their temperature. Here we discuss a trick that allows us to calculate this temperature in a simple way. The trick is not justified here but it really does work. The idea is that, since black holes are thermal objects, they must have a periodicity in Euclidean time $\tau \sim \tau + \beta$. To find the temperature $T = 1/\beta$, we take the near horizon limit of the geometry and ask that it is smooth. More specifically we ask that there is no conical defect and it is enough to fix the periodicity.

A black hole is in general described by a metric of the form

$$ds^2 = -f(r)\, dt^2 + \frac{dr^2}{f(r)} + r^2\, d\Omega^2 \tag{13}$$

where $f(r)$ has a zero at the horizon r_0. The Euclidean version of this is obtained as usual by putting $t = -i\tau$ and the effect is just to cancel the minus sign in front of the time component of the metric. To go to the near horizon limit, we write $r = r_0 + \epsilon$ and consider ϵ to be small. We also ignore the angular part since it doesn't play a role in this story. In that case $f(r) \approx f'(r_0)\epsilon$ and then

$$ds^2 \approx \epsilon f'(r_0)\, d\tau^2 + \frac{d\epsilon^2}{\epsilon f'(r_0)} . \tag{14}$$

To recover a more familliar geometry, we define the following variables:

$$R \equiv 2\sqrt{\frac{\epsilon}{f'(r_0)}} \qquad\qquad \Theta \equiv \frac{\tau f'(r_0)}{2}\,, \tag{15}$$

such that the metric becomes simply flat space in polar coordinates

$$ds^2 \approx dR^2 + R^2\, d\Theta^2\,. \tag{16}$$

The important point here is that for this geometry to be smooth we need Θ to be periodic with period 2π. Given that Θ is related to τ, this periodicity requirement can be translated to a period of $4\pi/f'(r_0)$ for τ. This means that

$$T_{BH} = \frac{1}{\beta} = \frac{f'(r_0)}{4\pi}\,. \tag{17}$$

As an example, for the Schwarzschild black hole $f(r) = 1 - \frac{2M}{r}$ and $r_0 = 2M$. This leads to $T = \frac{1}{8\pi M}$, which agrees with Hawking's result.